3/1/2012: First Midterm Exam

Your Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- Except for multiple choice problems, give computations.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points) No justifications are needed.

1) T F — 1 is the only root of the log function on the interval $(0, \infty)$.

Solution:
Yes, log is monotone and has no other root.

2) T F — $\exp(\log(5)) = 5$, if log is the natural log and $\exp(x) = e^x$ is the exponential function.

Solution:
Yes, exp is the inverse of log.

3) T F — The function $\cos(x)+\sin(x)+x^2$ is continuous everywhere on the real axes.

Solution:
It is the sum of three functions for which we know it to be true.

4) T F — The function $\sec(x) = 1/\cos(x)$ is the inverse of the function $\cos(x)$.

Solution:
No, it is $\arccos(x)$ which is the inverse.

5) T F — The Newton method allows to find the roots of any continuous function.

Solution:
The function needs to be differentiable.

6) T F — $\sin(3\pi/2) = -1$.

Solution:
Draw the circle. The angle $3\pi/2$ corresponds to 270 degrees. The sin is the y value and so -1.

7) T F — If a function f is continuous on $[0, \infty)$, then it has a global maximum on this interval.

Solution:
$\exp(x)$ is a counter example. We would need a finite interval.

8) | T | **F** (marked) |

The reciprocal rule assures that $d/dx(1/g(x)) = -1/g(x)^2$.

Solution:
We have no g'

9) | **T** (marked) | F |

If $f(0) = g(0) = f'(0) = g'(0) = 0$ and $g''(0) = f''(0) = 1$, then $\lim_{x \to 0}(f(x)/g(x)) = 1$

Solution:
This is a consequence of l'Hopital's rule when applied twice.

10) | **T** (marked) | F |

An inflection point is a point, where the function $f''(x)$ changes sign.

Solution:
This is a definition.

11) | **T** (marked) | F |

If $f''(3) > 0$ then f is concave up at $x = 3$.

Solution:
The slope of the tangent increases which produces a concave up graph. One can define concave up with the property $f''(x) > 0$

12) | T | **F** (marked) |

The intermediate value theorem assures that a continuous function has a maximum on a finite interval.

Solution:
The intermediate value theorem deals with roots.

13) | **T** (marked) | F |

We can find a value b and define $f(0) = b$ such that the function $f(x) = (x^6 - 1)/(x^3 - 1)$ is continuous everywhere.

Solution:
We divide by zero at $z = 1$.

14) T F Single roots of the second derivative function f'' are inflection points.

Solution:
Indeed, f'' changes sign there.

15) T F If the second derivative $f''(x)$ is negative and $f'(x) = 0$ then f has a local maximum at x.

Solution:
This is part of the second derivative test

16) T F The function $f(x) = [x]^3 = x(x+h)(x+2h)$ satisfies $Df(x) = 3[x]^2 = 4x(x+h)$, where $Df(x) = [f(x+h) - f(x)]/h$.

Solution:
Yes, this is a cool property of the polynomials $[x]^n$ but only if $[x]^3 = x(x-h)(x-2h)$ is chosen.

17) T F The quotient rule is $d/dx(f/g) = (fg' - f'g)/g^2$.

Solution:
This is an important rule to know but the sign is off!

18) T F The chain rule assures that $d/dx f(g(x)) = f'(g(x))f'(x)$.

Solution:
This is not true. We have $f'(g(x))g'(x)$.

19) T F With $Df(x) = f(x+1) - f(x)$, we have $D2^x = 2^x$.

Solution:
So cool.

20) T F Hôpital's rule applied to the function $f(x) = \text{sinc}(x) = \sin(x)/x$ gives us the fundamental theorem of trigonometry.

Solution:
Yes, this is the ultimate way to verify that.

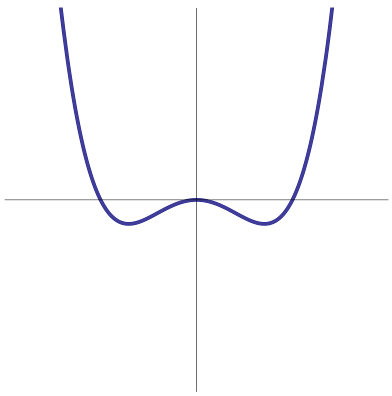
Problem 2) Matching problem (10 points) No justifications are needed.

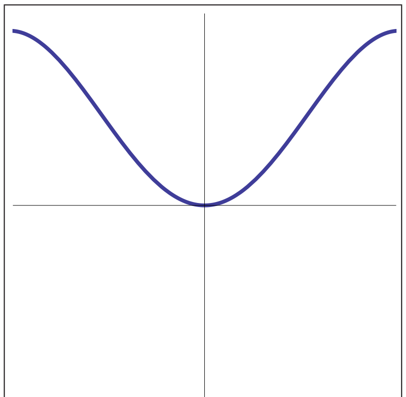
Match the functions with the graphs.

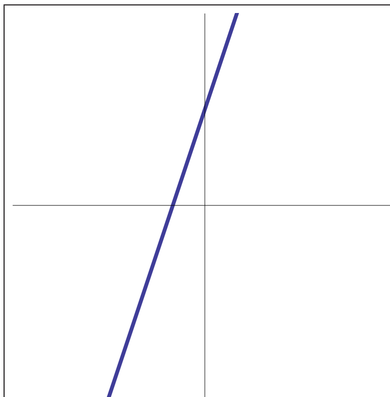
Function	Enter 1-9
$1/(1+x^2)$	
$\cot(2x)$	
$3x+1$	

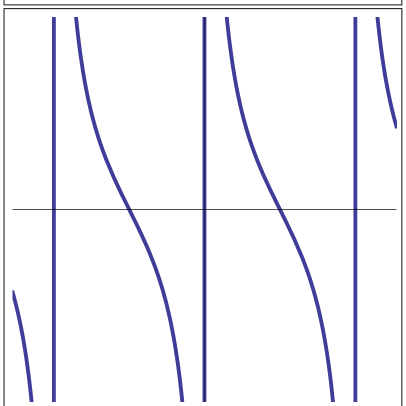
Function	Enter 1-9
$x\sin(x)$	
$\exp(-x)$	
$\log(\|x\|)$	

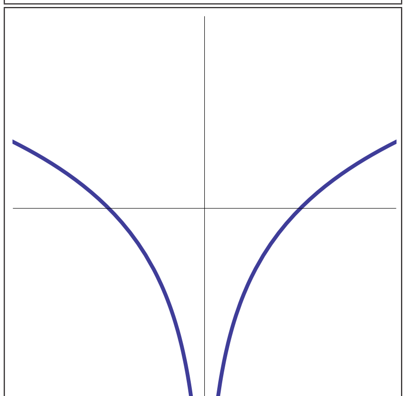
Function	Enter 1-9
$\text{sign}(x)$	
x^4-x^2	
$-x^2$	

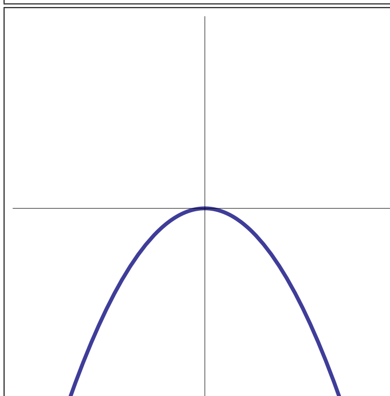

1)

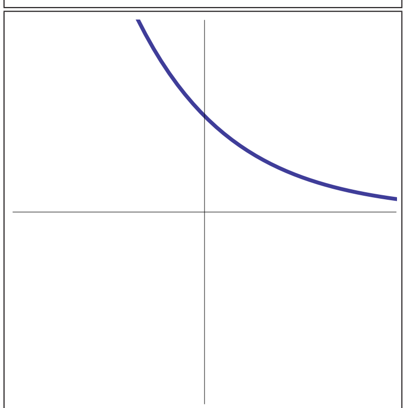

2)

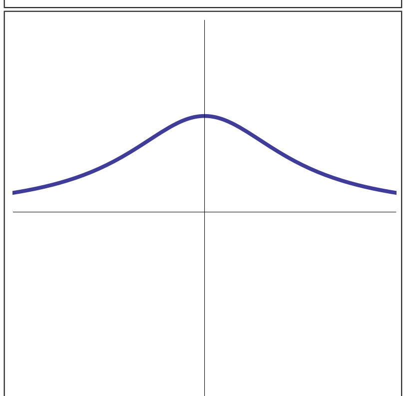

3)

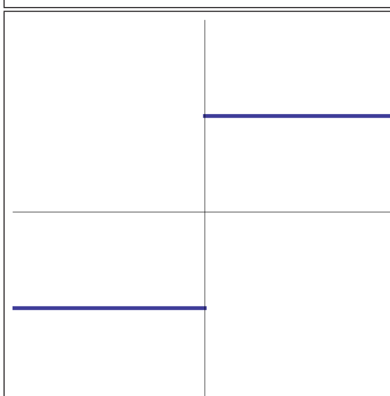

4)


5)


6)


7)


8)

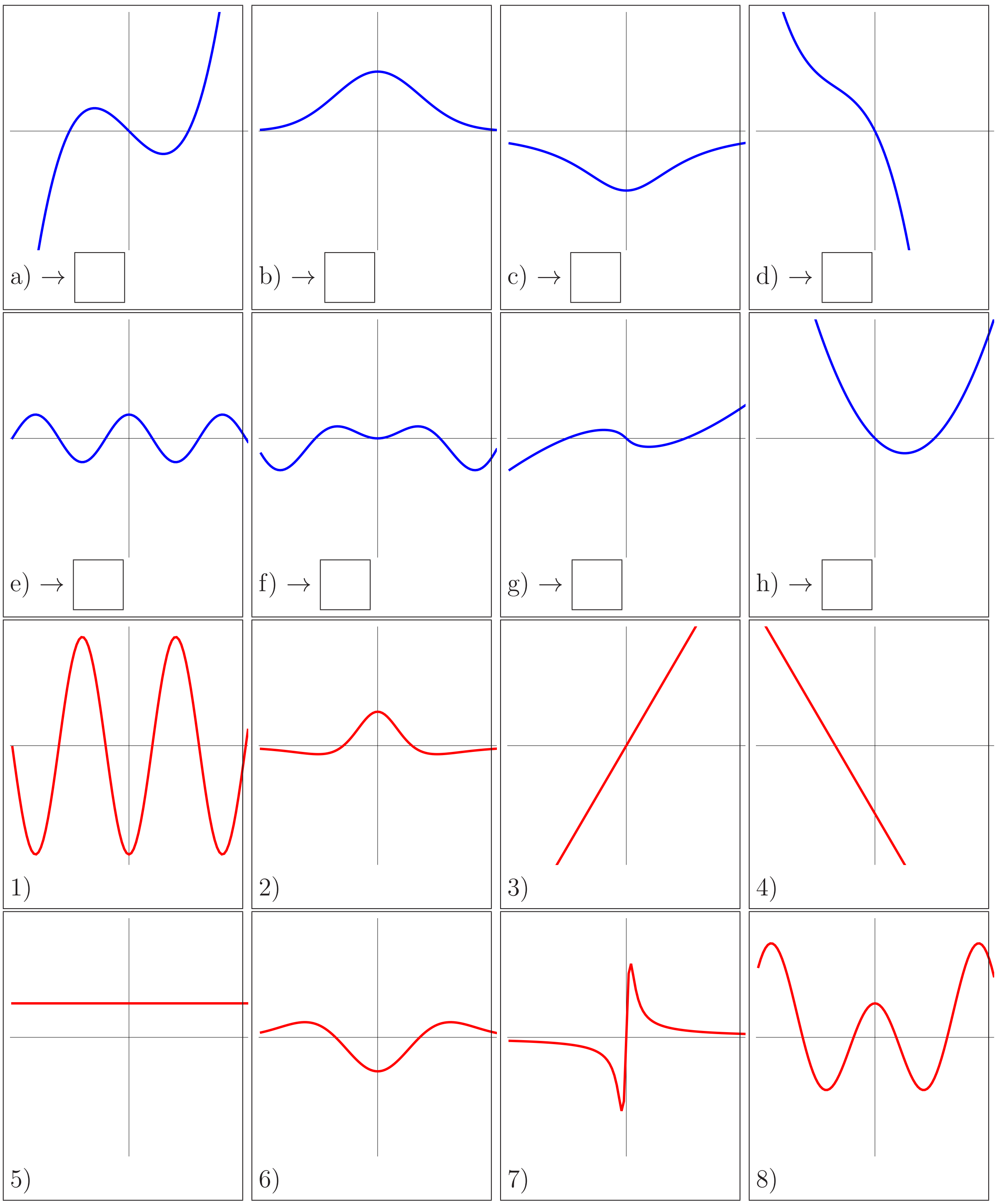

9)

Solution:

8)	2)	9)
4)	7)	1)
3)	5)	6)

Problem 3) Matching problem (10 points) No justifications are needed.

Match the graph of the functions f in $a) - h)$ with the second derivatives f'' in 1)-8).



Solution:
3) 6) 2) 4)
1) 8) 7) 5)

Problem 4) Continuity (10 points)

Decide whether the function can be healed at the given point in order to be continuous everywhere on the real line. If the function can be extended to a continuous function, give the value at the point.

a) (2 points) $f(x) = \frac{(x^3-8)}{(x-2)}$, at $x = 2$

b) (2 points) $f(x) = \sin(\sin(1/x)) - \tan(x)$, at $x = 0$

c) (2 points) $f(x) = \frac{\cos(x)-1}{x^2}$, at $x = 0$

d) (2 points) $f(x) = (\exp(x) - 1)/(\exp(5x) - 1)$, at $x = 0$

e) (2 points) $f(x) = \frac{(x-1)}{x}$, at $x = 0$

Solution:
a) We can use Hôpital's rule to see that the limit is $\lim_{x \to 2} 3x^2/1 = 12$.
b) There is no way that we can save the oscillatory singularity.
c) Apply Hopital twice to see that the limit is $-1/2$.
d) Apply l'Hopital to see that the limit is $\lim_{x \to 0} e^x/(5e^{5x}) = 1/5$.
e) This can not be saved at $x = 0$. There is a pole there.

Problem 5) Chain rule (10 points)

In the following cases, we pretend not to know the formula for the derivative of log or arctan and again recover it using the chain rule.

b) (2 points) Rederive the derivative of the square root function $\text{sqrt}(x) = \sqrt{x}$ by differentiating

$$(\text{sqrt}(x))^2 = x$$

and solving for $\text{sqrt}'(x)$.

b) (4 points) Rederive the derivative of the logarithm function $\log(x)$ by differentiating

$$\exp(\log(x)) = x$$

and solving for $\log'(x)$.

c) (4 points) Rederive the formula for the derivative of the arctan function $\arctan(x)$ by differentiating the identity

$$\tan(\arctan(x)) = x$$

and using $1 + \tan^2(x) = 1/\cos^2(x)$ to solve for $\arctan'(x)$.

Solution:
a) Differentiate $(\sqrt{x})^2 = x$ to get $2\sqrt{x}\frac{d}{dx}\sqrt{x} = 1$ so that $\frac{d}{dx}\sqrt{x} = 1/(2\sqrt{x})$.
b) Differentiate $\exp(\log(x)) = x$ to get $\exp(\log(x))\log'(x) = 1$ and solve for $\log'(x) = 1/\exp(\log(x)) = 1/x$.
c) Differentiate $\tan(\arctan(x)) = x$ to get $\sec^2(\arctan(x))\arctan'(x) = 1$ and use $\sec^2(x) = 1 + \tan^2(x)$ to see $\arctan'(x) = 1/(1 + x^2)$.

Problem 6) Derivatives (10 points)

Find the derivatives of the following functions:

a) (2 points) $f(x) = \frac{5\sin(x^6)}{x}$ for $x > 0$

b) (2 points) $f(x) = \tan(x^2) + \cot(x^2)$ for $x > 0$

c) (2 points) $f(x) = \frac{1}{x} + \log(x^2)$ for $x > 0$

d) (2 points) $f(x) = x^6 + \sin(x^4)\log(x)$ for $x > 0$

e) (2 points) $f(x) = \log(\log(x))$ for $x > 1$

Solution:
Use the product, quotient and chain rule.
a) $30x^4\cos(x^6) - \frac{5\sin(x^6)}{x^2}$
b) $2x\sec^2(x^2) - 2x\csc^2(x^2)$
c) $\frac{2}{x} - \frac{1}{x^2}$.
d) $6x^5 + \frac{\sin(x^4)}{x} + 4x^3\log(x)\cos(x^4)$.
e) $\frac{1}{x\log(x)}$.

Problem 7) Limits (10 points)

Find the limits $\lim_{x\to 0} f(x)$ for the following functions f at $x = 0$ or state that the limit does not exist. State the tools you are using.

a) (2 points) $f(x) = x^2 + x + \sin(1 - \cos(x))$

b) (2 points) $f(x) = \frac{x^3}{\sin(x^3)}$

c) (2 points) $f(x) = x^3/\sin(x)^2$

d) (2 points) $f(x) = x^3 + \text{sign}(x)$

e) (2 points) $f(x) = \cos(x^4) + \cos(\frac{1}{x})$

Solution:
a) There is no problem at all at $x = 0$. We can just plug in the value 0 and get $\boxed{0}$.
b) We know that $x/\sin(x)$ has the limit 1. Therefore the limit is $\boxed{1}$ too.
c) We can write this as $x(x^2/\sin^2(x))$. The right part has the limit 1 by the fundamental theorem of trigonometry. Therefore the limit is $\boxed{0}$.
d) This function has no limit at $x = 0$ because $\text{sign}(x)$ has a jump singularity at 0.
e) Also this function has no limit at $x = 0$ because $\cos(1/x)$ is a devil comb at 0 with an oscillatory discontinuity.

Problem 8) Extrema (10 points)

In the following problem you can ignore the story if you like and proceed straight go to the question:

Story: a cone shaped lamp designed in 1995 by **Verner Panton** needs to have volume $\pi r^2 h = \pi$ to be safe. To minimize the surface area $A = \pi r\sqrt{h^2 + r^2}$, we minimize the square A^2 and so $\pi^2 r^2(h^2 + r^2)$. From the volume assumption, we get $r^2 = 1/h$ so that we have to minimize $\pi^2/h(h^2 + 1/h)$.

Which height h minimizes the function

$$f(h) = h + \frac{1}{h^2} \ ?$$

Use the second derivative test to check that you have a minimum.

Solution:
The derivative is $f'(h) = 1 - 2/h^3 = 0$. This is zero at $2 = h^3$ which means $\boxed{h = 2^{1/3}}$. The second derivative is $f''(h) = 6/h^4$ which is positive at h. We have a **local minimum**.

Problem 9) Global extrema (10 points)

An investment problem leads to the profit function

$$f(x) = x - 2x^2 + x^3 ,$$

where $x \in [0, 2]$. Find the local and global maxima and minima of f on this interval and use the second derivative test.

Solution:
First find the critical points: $f'(x) = 1 - 4x + 3x^2 = 0$ for $\boxed{x = 1/3}$ and $\boxed{x = 1}$. The second derivative is $f''(x) = -4 + 6x$ which is -2 for $x = 1/3$ and 2 for $x = 1$. Therefore, $1/3$ is a local minimum and 1 is a local maximum. In order to find the global extrema, we have to evaluate the function also at the boundary $0, 2$ and compare the values

$$f(1/3) = 3/27, f(1) = 0, f(2) = 2, f(0) = 0 .$$

The global maximum is at $\boxed{x = 2}$, the global minimum is attained at the two points $x = 0$ and $x = 1$.